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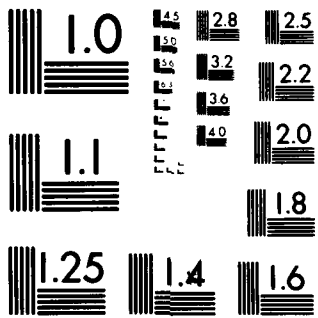
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ON OPTIMUM INSPECTION SCHEDULES

by

S. PATRICK KOH

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# ON OPTIMUM INSPECTION SCHEDULES<sup>1</sup>

by  
S. Patrick Koh

## Abstract

This paper treats the problem of determining the minimax inspection schedule for detecting failure of a component or system when inspections have a cost and cost of failure is proportional to the length of time between failure and detection. The minimaxing is done with respect to all failure distributions having a given mean.

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<sup>1</sup>This research forms part of a doctoral dissertation to be submitted for approval by the Department of Industrial Engineering and Operations Research, Columbia University. The research was carried out under the supervision of Professor Cyrus Derman.

## ON OPTIMUM INSPECTION SCHEDULES

### 1. Introduction

Consider a certain component of an operating system. The system must operate for  $T$  units of time.  $T$  may be finite (the finite horizon case) or infinite (the infinite horizon case). The component can fail at a random time  $Y$ . However, if it fails there is a cost incurred that is proportional to the time between failure and its detection. Failure can only be detected if the component is inspected. However, there is a cost for each inspection. Our interest concentrates on the problem of determining an inspection schedule  $x$  that reconciles the two types of cost. Whatever occurs subsequent to the detection of a failure or to the completion of the system's mission at time  $T$  is not considered.

In particular, if  $x = (x_0, x_1, \dots, x_{n+1})$ ,  $x_0 = 0$ ,  $x_{n+1} = T$  in the finite horizon case, or  $x = (x_0, x_1, \dots)$ ,  $x_0 = 0$  in the infinite horizon case is a given schedule of inspection,  $F$  is the distribution function of  $Y$ ,  $c$  is the cost of an inspection, and  $v$  is the cost per unit time of an undetected component failure, then the expected cost associated with the schedule is

$$C(x, F) = \sum_{r=0}^n \int_{x_r}^{x_{r+1}} \{(r+1)c + v(x_{r+1} - t)\} dF + (n+1)c(1-F(T))$$

, for  $T < \infty$ ,  $n < \infty$

$$= \sum_{r=0}^{\infty} \int_{x_r}^{x_{r+1}} \{(r+1)c + v(x_{r+1} - t)\} dF(t)$$

, for  $T = \infty$ ,  $n = \infty$ .

When  $F$  is not completely known the problem is to find  $x$  to minimize  $U(x)$  where  $U(x) = \sup_{F \in \mathcal{C}} C(x, F)$ , and  $\mathcal{C}$  is the class of possible distribution functions. Assuming  $T < \infty$ , Derman [2], originally, considered this problem for the class  $\mathcal{C}$  of all distribution functions for non-negative random variables, allowing for the possibility that detection of a failure at an inspection is uncertain. Explicit formula for the optimal schedules, called minimax schedules, were obtained. Roeloffs [5], assuming certain detection, obtained minimax schedules for the case when  $\mathcal{C}$  is the class of distributions having a known quantile. Kander & Raviv [4] assumed Roeloffs's case with the added assumption that all distributions in  $\mathcal{C}$  have increasing failure rates, Beichelt [1] generalized Derman's results to the case that the failure costs were, rather than being linear, an increasing function of the time between failure and detection. Beichelt, also, considered, in the infinite horizon case, with increasing failure costs, the minimax schedule when  $\mathcal{C} = \mathcal{C}_\mu$  consists of all non-negative distributions  $F$  with a given mean  $\mu$ . In this case, he proves that an optimal schedule has equal intervals.

We consider for the finite horizon case the problem of determining the minimax schedule when  $\mathcal{C} = \mathcal{C}_\mu$ . In Derman [2], the observation that  $\mathcal{C}$  can be reduced to  $\mathcal{C}_1^1$ , the class of degenerate distributions (or one point distributions), is exploited. Analogously, we employ a theorem due to Hoeffding [3], that enables us to reduce  $\mathcal{C}_\mu$  to  $\mathcal{C}_\mu^2$ , the class of all two-point

distributions with mean  $\mu$ . Results are obtained that permit the numerical determination of the minimax schedule. We also obtain Beichelt's infinite horizon schedules by the limit of minimax schedules as  $T \rightarrow \infty$ .

## 2. The minimax schedule where only the mean of $F$ is known.

We are interested in the problem of determining the minimax schedule when only the mean of  $F$  is known. For the infinite horizon case, Beichelt [1] showed that the minimax schedule is a strictly periodic schedule with the equal interval space  $\delta$ ,  $\delta = \sqrt{\frac{\mu C}{v}}$ .

We focus on the problem for the finite horizon case. We start with this problem by considering the supremum expected cost for a given schedule  $x$ , that is  $\sup_{F \in \mathcal{L}_\mu^2} C(x, F)$ . Hoeffding [3] proved  $\sup_{F \in \mathcal{L}_\mu^2} C(x, F) = \sup_{F \in \mathcal{L}_\mu^1} C(x, F)$ . In section 2.1, we

show

$$\sup_{\substack{0 \leq i \leq m \\ m+1 \leq j \leq n}} \{G_i(x), G_{ij}(x)\} = \sup_{F \in \mathcal{L}_\mu^2} \{C(x, F)\}$$

where  $x_m < \mu$ ,  $x_{m+1} \geq \mu$ ,

$$G_r(x) = (r+1)C + v(x_{r+1} - x_r)$$

,  $r=0, 1, 2, \dots, n$ ,  $x_0=0$ ,  $x_{n+1} = T$

and

$$G_{ij}(x) = G_i(x) \frac{x_j - \mu}{x_j - x_i} + G_j(x) \frac{\mu - x_i}{x_j - x_i}.$$



In section 2.2., we prove a key theorem, which gives a necessary condition of the minimax schedule, that is

$$(2.1) \quad \frac{G_{r+1}(x) - G_r(x)}{x_{r+1} - x_r} = Z$$

$r=0,1,2,\dots,n$  for some  $Z \geq 0$ . We call the schedule satisfying (2.1) a schedule with equal average increment (SEAI). The supremum expected cost of a SEAI has a simple form,  $G_0(x) + Z\mu$ .

In section 2.3., we convert this problem to a problem of determining the minimal point of a continuous function with a single variable. Under some certain conditions, the function is piecewise convex.

In section 2.4., we show that the minimax schedule for the infinite horizon case is the limit of the minimax schedule as  $T \rightarrow \infty$ .

In section 2.5., when  $T < \infty$ , an algorithm for computing minimax schedule is given.

## 2.1. Supremum expected cost of an inspection schedule.

Consider a certain component of an operating system. The component can fail at a random time  $Y$ ; occasional inspection is necessary to determine when it does fail. The cost of a failure,  $vt$ , where  $t$  is the time between the failure and its inspection, and  $v$  is the cost per unit of  $t$ , can be reduced by frequent inspection. However, there is a cost  $c$  for each inspection, so the number of inspections must be kept small. The best compromise between these two conflicting requirements in the sense of minimizing expected total costs is called the optimum inspection schedule. In a finite horizon problem we

assume the system will stop operating when a stipulated time  $T$  is reached or when a failed component is detected.

In this section, we will derive a formula for supremum expected total cost for any inspection schedule assuming the mean of  $Y$  is given.

Definitions:

$\mu$  is the mean of  $Y$ ;

$\mathcal{Q}$  is the class of all distribution functions for nonnegative random variables.

$\mathcal{Q}_\mu$  is the subset of  $\mathcal{Q}$  that have expected lifetime  $\mu$ ;

$\mathcal{Q}_\mu^2$  is the subset of  $\mathcal{Q}_\mu$  that have exactly two points of increase;

$x = (x_0, x_1, \dots, x_n, x_{n+1})$  denotes an inspection schedule (short: schedule), where  $x_0 = 0$ ,  $x_{n+1} = T$ , and  $x_i < x_{i+1}$ ,  $i = 0, 1, 2, \dots, n$ ;

$m$  is the index of an inspection in  $x$  such that  $x_m < \mu$ ,  $x_{m+1} \geq \mu$ ;

$C(x, F)$  is the expected total cost of  $x$  if  $F$  is the c.d.f. of  $Y$ ;

The supremum expected cost of  $x$  is  $u(x) = \sup_{F \in \mathcal{Q}} C(x, F)$ .

If  $y_1, y_2$  denotes two increasing points of the c.d.f.,  $F(y_1, y_2)$ , of  $Y$ , which is a member of  $\mathcal{Q}_\mu^2$  then the probability function of  $Y$  is

$$P(Y = y_1) = \frac{y_2 - \mu}{y_2 - y_1}$$

$$P(Y = y_2) = \frac{\mu - y_1}{y_2 - y_1}$$

, where  $0 < y_1 < \mu < y_2 < \infty$ .

Proposition 2.1. If  $x$  is a given schedule with  $x_{m+1} \leq T$  and  $F(y_1, y_2)$  is a c.d.f. in  $\mathcal{G}_\mu^2$  with  $\mu < y_2 \leq x_{m+1}$  then  $C(x, F(y_1, y_2)) < u(x)$ .

Proof.

$$\begin{aligned} & C(x, F(y_1, y_2)) \\ &= [(i+1)C + V(x_{i+1} - y_1)] \frac{y_2 - \mu}{y_2 - y_1} + [(m+1)C + V(x_{m+1} - y_2)] \frac{\mu - y_1}{y_2 - y_1} \end{aligned}$$

, if  $x_i < y_1 \leq x_{i+1}$  and  $\mu < y_2 \leq x_{m+1}$ .

By taking partial derivative with respect to  $y_2$ , we have

$$\begin{aligned} & \frac{\partial}{\partial y_2} C(x, F(y_1, y_2)) \\ &= \frac{\mu - y_1}{(y_2 - y_1)^2} [(i-m)C + V(x_{i+1} - x_{m+1})] \leq 0 \end{aligned}$$

which shows that  $C(x, F(y_1, y_2))$  is a strictly decreasing function on  $y_2$ ; hence, we have

$$\begin{aligned} & C(x, F(y_1, y_2)) \\ & \leq C(x, F(y_1, \mu)) \\ &= (m+1)C + V(x_{m+1} - \mu) \\ & < (m+1)C + V(x_{m+1} - x_m) \\ &= \lim_{y_2 \rightarrow \infty} C(x, F(x_m, y_2)) \\ & \leq U(x) \end{aligned}$$

Q.E.D.

Proposition 2.2.  $U(x) = \sup_{F \in \mathcal{G}_\mu^2} C(x, F)$

Proof. Let  $\mathcal{C}_\mu^* = \{F \in \mathcal{C}_\mu \mid F \text{ is a step function}\}$  and  $d(F, G) = \sup_t |F(t) - G(t)|$ ; theorem 2.2. in Hoeffding [3] states

$$(2.1.1.) \quad \sup_{F \in \mathcal{C}_\mu^2} C(x, F) = \sup_{F \in \mathcal{C}_\mu^*} C(x, F) = \sup_{F \in \mathcal{C}_\mu} C(x, F)$$

if (A)  $C(x, F)$  is a continuous function on  $\mathcal{C}_\mu$  for  $x$  fixed, in the sense of  $d(.,.)$ , and (B) for any  $F \in \mathcal{C}_\mu$ , there is a sequence  $\{F_n\} \subseteq \mathcal{C}_\mu^*$  such that  $\lim_{n \rightarrow \infty} F_n = F$  in the sense of  $d(.,.)$ .

The satisfaction of A and B are easily verified.

Definitions:

$$\begin{aligned} f_{ij}(y_1, y_2) \\ &= C(x, F(y_1, y_2)) \\ &= g_i(y_1) \frac{y_2 - \mu}{y_2 - y_1} + g_j(y_2) \frac{\mu - y_1}{y_2 - y_1} \end{aligned}$$

, where  $x_m < \mu \leq x_{m+1}$ ,  $1 \leq i \leq m$ ,  $m+1 \leq j \leq n+1$ ,

$y_1 \in (x_i, x_{i+1}]$ ,  $y_2 \in (x_j, x_{j+1}]$ , and

$$g_r(y) = (r+1)C + V(x_{r+1} - y) \quad \text{if } y \in (x, x_{r+1}] \text{ and}$$

$$r = 0, 1, \dots, n$$

$$= (n+1)C \quad \text{if } y > T \text{ and } r = n+1 ;$$

$$G_{ij}(x) = f_{ij}(x_i +, x_j +) ;$$

$$G_r(x) = g_r(x_r +).$$

Theorem 2.1. For any schedule  $x$ ,

$$u(x) = \sup_{0 \leq i \leq m} \{G_{ij}(x), G_i(x)\}.$$

$$0 \leq i \leq m$$

$$m+1 \leq j \leq n$$

Proof. The partial derivatives with respect to  $y_1$  and  $y_2$  on  $f_{ij}(y_1, y_2)$  respectively are

$$\frac{\partial f_{ij}(y_1, y_2)}{\partial y_1}$$

$$(2.1.3) = \frac{y_2^{-\mu}}{(y_2 - y_1)^2} [(i-j)C + V(x_{i+1} - x_{j+1})] < 0, \text{ if } j \leq n,$$

$$(2.1.4) = \frac{y_2^{-\mu}}{(y_2 - y_1)^2} [(i-n)C + V(x_{i+1} - y_2)] < 0, \text{ if } j = n+1.$$

$$\frac{\partial f_{ij}(y_1, y_2)}{\partial y_2}$$

$$(2.1.5) = \frac{\mu - y_1}{(y_2 - y_1)^2} [(i-j)C + V(x_{i+1} - x_{j+1})] < 0, \text{ if } j \leq n,$$

$$(2.1.6) = \frac{\mu - y_1}{(y_2 - y_1)^2} [(i-n)C + V(x_{i+1} - y_1)] < 0, \text{ if } j = n+1.$$

If the value of (2.1.6) is negative then  $f_{in+1}(y_1, y_2) < G_{in}(x)$ , together with (2.1.1), proposition 2.1., (2.1.3), (2.1.4), (2.1.5), we have

$$U(x) = \sup_{\substack{0 \leq i \leq m \\ m+1 \leq j \leq n}} \{G_{ij}(x)\}$$

If the value of (2.1.6) is nonnegative then, we have

$$f_{in+1}(y_1, y_2) \leq \lim_{y_2 \rightarrow \infty} f_{in+1}(y_1, y_2) = g_i(y_1) \leq G_i(x)$$

which implies

$$U(x) = \sup_{\substack{0 \leq i \leq m \\ m+1 \leq j \leq n}} \{G_{ij}(x), G_i(x)\}$$

Q.E.D.

2.2. The minimax schedule is a schedule with equally average increments

Definitions:

$i$  denotes the index of an inspection of a schedule  $x$  such that  $0 \leq i \leq m$ ;

$j$  denotes the index of an inspection of a schedule  $x$  such that  $m+1 \leq j \leq n$ .

Proposition 2.3.  $x$  is a given schedule. If  $j \neq m+1$ , or  $j=m+1$  but  $i \neq m$ , then

$G_{ij}(x)$  is an increasing function on  $x_{i+1}, x_{j+1}$  and decreasing function on  $x_i, x_j$ .

If  $j = m+1$  and  $i=m$ , then

$G_{m(m+1)}(x)$  is a decreasing function on  $x_m$ .

Proof.

If  $j \neq m+1$ , or  $j = m+1$  but  $i \neq m$  then

$$\frac{\partial G_{ij}}{\partial x_i} = \frac{x_j - \mu}{(x_j - x_i)^2} \{ (i-j)C - V(x_{j+1} - x_{i+1}) \} < 0$$

$$\frac{\partial G_{ij}}{\partial x_{i+1}} = V \frac{x_j - \mu}{x_j - x_i} > 0$$

$$\frac{\partial G_{ij}}{\partial x_j} = \frac{\mu - x_i}{(x_j - x_i)^2} \{ (i-j)C - V(x_{j+1} - x_{i+1}) \} < 0$$

$$\frac{\partial G_{ij}}{\partial x_{j+1}} = V \frac{\mu - x_i}{x_j - x_i} > 0.$$

If  $j = m+1$  and  $i = m$  then

$$\frac{\partial G_m(m+1)}{\partial x_m}$$

$$= \frac{x_{m+1} - \mu}{(x_{m+1} - x_m)^2} \{-C - V(x_{m+2} - x_{m+1})\} < 0$$

Q.E.D.

Proposition 2.4. For any schedule  $x$ ,  $G_{ij}(x) < G_i(x)$  if and only if  $G_j(x) < G_i(x)$ .

Proof.  $G_{ij}(x) = G_i(x) + \frac{G_j(x) - G_i(x)}{x_j - x_i} (\mu - x_i)$

$$< G_i(x)$$

$$\Leftrightarrow G_j(x) < G_i(x), \text{ since } \mu - x_i > 0$$

Q.E.D.

Proposition 2.5. If  $x$  is the minimax schedule and  $\{j\} \mid n \geq j \geq m+1$  is nonempty set, then  $u(x) = \sup_{i,j} \{G_{ij}(x)\}$ .

Proof. If  $u(x) = G_{i_0}(x)$  for some  $i_0 \leq m$ , and  $G_{i_0}(x) > G_{i_0 j}(x)$  for all  $j \in \{j\} \mid n \geq j \geq m+1$ , then, by nonemptiness of  $\{j\} \mid n \geq j \geq m+1$  and proposition 2.4., we have  $G_{m+1}(x) < G_{i_0}(x)$ . Hence, we can find a sufficiently small  $\epsilon > 0$  and the corresponding schedule

$$x^\epsilon = (0, \dots, x_i - \frac{\epsilon}{2(m-i+1)}, \dots, x_{m+1} - \epsilon, x_{m+2}, \dots, x_n) \text{ such}$$

that

$$(2.2.2) \quad G_{m+1}(x) < G_{m+1}(x^\epsilon) < G_{i_0}(x).$$

Since  $x_{i+1}^\epsilon - x_i^\epsilon < x_{i+1} - x_i$ ,  $i=0,1,\dots,m$  and  $x_{j+1}^\epsilon - x_j^\epsilon = x_{j+1} - x_j$ ,

$j = m+2, \dots, n$ , we also have

$$(2.2.3) \quad G_i(x^\epsilon) < G_i(x) \leq G_{i_0}(x), \quad i=0,1,\dots,m \text{ and}$$

$$(2.2.4) \quad G_j(x^\epsilon) = G_j(x) < G_{i_0}(x), \quad j = m+2, \dots, n.$$

Case 1.  $x_{m+1} > \mu$ .

If  $\epsilon$  is sufficiently small such that  $x_{m+1} - \epsilon > \mu$ , then, by (2.2.2), (2.2.3), (2.2.4), proposition 2.4. and theorem 2.1, we have

$$(2.2.5) \quad U(x^\epsilon) = \sup_{i,j} \{G_{ij}(x^\epsilon), G_i(x^\epsilon)\} \\ < G_{i_0}(x) = U(x).$$

Case 2.  $x_{m+1} = \mu$

In this case,  $x_{m+1} - \epsilon < \mu$ . (2.2.5.) becomes

$$U(x^\epsilon) = \sup_{\substack{0 \leq i \leq m+1 \\ m+2 \leq j \leq n}} \{G_{ij}(x^\epsilon), G_i(x^\epsilon)\} \\ < G_{i_0}(x) \\ = U(x).$$

Both case 1 and case 2 lead to a contradiction; thus,

$$U(x) = \sup_{i,j} \{G_{ij}(x)\}.$$

Q.E.D.

Definitions:

If  $x$  is a schedule,  $G_j(x) - G_i(x)$  is called an increment with respect to  $(i,j)$  and  $\Delta x(j,i) = \frac{G_j(x) - G_i(x)}{x_j - x_i}$  is called



an average increment with respect to  $(i,j)$ . If  $x$  is a schedule such that for some  $Z$   $\Delta x(r+1,r) = Z$ ,  $r=0,1,\dots,n$  then  $x$  is called a schedule with equal average increment (SEAI).

Proposition 2.6. If for a schedule  $x$ , there exists  $i_0$  and  $j_0$  such that

$$(2.2.6) \quad G_{i_0 m+1}(x) = \dots = G_{i_0 n}(x) \text{ and}$$

$$(2.2.7) \quad G_{0 j_0}(x) = \dots = G_{m j_0}(x)$$

then  $x$  is a SEAI.

Proof. For convenience, let  $G_{ij} = G_{ij}(x)$ ,  $G_r = G_r(x)$ ,

$\Delta(i,j) = \Delta x(i,j)$ . Since

$$\begin{aligned} G_{ij} &= G_i + \Delta(j,i) (\mu - x_i) \\ &= G_j - \Delta(j,i) (x_j - \mu) \end{aligned}$$

for any schedule  $x$  and any existing pair  $(i,j)$  we have,

by (2.2.6),

$$\begin{aligned} &\Delta(j, i_0) - \Delta(j+1, i_0) \\ &= \frac{G_{i_0 j} - G_{i_0}}{\mu - x_{i_0}} - \frac{G_{i_0 j+1} - G_{i_0}}{\mu - x_{i_0}} = 0 \end{aligned}$$

,  $j = m+1, \dots, n$ .

If  $j_0 \neq m+1$  or  $x_{m+1} > \mu$ , then, by (2.2.7) and the same reason as above, we have

$$(2.2.8) \quad \Delta(j_0, i) - \Delta(j_0, i+1) = 0, \quad i=0,1,\dots,m-1. \quad \text{If } j_0 = m+1$$

and  $x_{m+1} = \mu$  then  $G_{im+1} = G_{m+1}$ ,  $i=0,\dots,m$ , which implies

$G_i = G_{m+1}$ ; thus, (2.2.8) still holds.

In other words, the following holds:

$$(2.2.9) \quad \Delta(m+1, i_0) = \dots = \Delta(n, i_0) \text{ and}$$

$$(2.2.10) \quad \Delta(j_0, 0) = \dots = \Delta(j_0, m).$$

Since (2.2.9) and (2.2.10) have the same term  $\Delta(j_0, i_0)$ , we have

$$(2.2.11) \quad \Delta(j_0, 0) = \Delta(j_0, 1) = \dots = \Delta(m+1, i_0) = \dots = \Delta(n, i_0).$$

Setting the expression of (2.2.11) equal to  $Z$ , by the definition of  $\Delta(\dots)$  and  $\Delta(j_0, i) = \Delta(j_0, i+1) = Z$ , we have

$$\begin{aligned} G_{i+1} - G_i &= G_{j_0} - G_i + G_{i+1} - G_{j_0} \\ &= (x_{j_0} - x_i)Z + (x_{i+1} - x_{j_0})Z \\ &= (x_{i+1} - x_i)Z. \end{aligned}$$

This implies

$$(2.2.12.a) \quad \Delta(i+1, i) = Z \text{ for } i=0, 1, \dots, m-1; \text{ similarly, by } \Delta(j+1, i_0) = \Delta(j, i_0) = Z \text{ we have}$$

$$(2.2.12.b) \quad \Delta(j+1, j) = Z \text{ for } j=m+1, \dots, n.$$

What remains to be shown is that the value of  $\Delta(m+1, m)$  is also  $Z$ . This follows from

$$\begin{aligned} Z &= \frac{G_{j_0} - G_{i_0}}{x_{j_0} - x_{i_0}} \\ &= \frac{(G_{j_0} - G_{j_0-1}) + (G_{j_0-1} - G_{j_0-2}) + \dots + (G_{m+1} - G_m)}{x_{j_0} - x_{i_0}} \\ &\quad + \frac{(G_m - G_{m-1}) + \dots + (G_{i_0+1} - G_{i_0})}{x_{j_0} - x_{i_0}} \end{aligned}$$

$$\begin{aligned}
&= \frac{Z(x_{j_0} - x_{j_0-1} + x_{j_0-1} - x_{j_0-2} \dots - x_{m+1}) + (G_{m+1} - G_m)}{x_{j_0} - x_{i_0}} \\
&\quad + \frac{Z(x_m - x_{m-1} + \dots - x_{i_0})}{x_{j_0} - x_{i_0}} \\
&= \frac{Z(x_{j_0} - x_{m+1}) + (G_{m+1} - G_m) + Z(x_m - x_{i_0})}{x_{j_0} - x_{i_0}} \\
&= Z + \frac{G_{m+1} - G_m - Z(x_{m+1} - x_m)}{x_{j_0} - x_{i_0}}
\end{aligned}$$

Cancelling  $Z$  from both sides of above equation we have

$$G_{m+1} - G_m = Z(x_{m+1} - x_m)$$

which implies  $\Delta(m+1, m) = Z$ . Together with (2.2.12.a) and (2.2.12.b) the proposition is proved.

Lemma 2.1. Suppose  $a, b, A, B$  are real numbers and  $B + b > 0$ ,  $B > 0$ ; we have

a)  $b > 0$ ;  $\frac{A + a}{B + b} < \frac{A}{B}$  if and only if  $\frac{a}{b} < \frac{A}{B}$ .

b)  $b < 0$ ;  $\frac{A + a}{B + b} < \frac{A}{B}$  if and only if  $\frac{a}{b} > \frac{A}{B}$ .

Proof.

$$a) \quad \frac{A+a}{B+b} < \frac{A}{B} \Leftrightarrow aB < bA$$

$$\Leftrightarrow a < \frac{bA}{B}$$

$$\Leftrightarrow \frac{a}{b} < \frac{A}{B} \quad (\text{by } b > 0)$$

$$b) \quad \frac{A+a}{B+b} < \frac{A}{B} \Leftrightarrow a < \frac{bA}{B}$$

$$\Leftrightarrow \frac{a}{b} > \frac{A}{B} \quad (\text{by } b < 0)$$

Q.E.D.

Definitions:

$$\psi_j(x) = \sup_{0 \leq i \leq m} \{G_{ij}(x)\}$$

$$\phi_i(x) = \sup_{m+1 \leq j \leq n} \{G_{ij}(x)\}.$$

Theorem 2.2. A minimax schedule  $x$  is a SEAI with some non-negative increment  $Z$ .

Proof. We prove the theorem under the following two cases:

Case 1: If  $\{j\}_{n \geq j \geq m+1} = \emptyset$  then  $x$  is a SEAI with average increment 0.

Case 2: If  $\{j\}_{n \geq j \geq m+1} \neq \emptyset$  then  $x$  is a SEAI with some nonnegative average increment  $Z$ .

Case 1. By theorem 2.1. and  $\{j\}_{n \geq j \geq m+1} = \emptyset$ , we have

$$U(x) = \sup_{0 \leq i \leq n} \{G_i(x)\}.$$

Suppose  $i_x$  is the largest index such that  $U(x) = G_{i_x}(x)$

and suppose  $i_x \neq n$ . Because  $G_{i_x}(x)$  is a continuous function in  $x_{i_x}$ ,  $x_{i_x+1}$ , and increasing in  $x_{i_x+1}$ , decreasing in  $x_{i_x}$  together with the assumption above, we can find a sufficiently small  $\epsilon > 0$  and a schedule

$$x^\epsilon = (x_1, x_2, \dots, x_{i_x}, x_{i_x+1} - \epsilon, \dots, x_n)$$

such that

$$G_{i_x+1}(x^\epsilon) < G_{i_x}(x) = U(x) \text{ still holds, and,}$$

(2.2.13)

$$G_{i_x}(x^\epsilon) < G_{i_x}(x) = U(x).$$

Thus  $U(x^\epsilon) \leq U(x)$ ; however  $x$  is minimax so  $U(x^\epsilon) = U(x)$

and  $i_x^\epsilon < i_x$ .

Repeating the above procedure at most  $i_x + 1$  times, we reach a schedule  $x^0$  such that  $U(x^0) = U(x)$  and  $i_x^0 = 0$ . Again, let  $\eta > 0$  be small enough with the corresponding schedule

$$x^* = (0, x_1 - \eta, x_2, \dots, x_n)$$

such that  $G_1(x^*) < G_0(x^0)$  and  $G_0(x^*) < G_0(x^0)$  then we have  $U(x^*) < U(x^0) = U(x)$ , a contradiction. Thus,  $i_x = n$ .

If there is a  $i_0$  such that  $G_{i_0}(x) < G_n(x) = U(x)$  then we can find a sufficiently small  $\epsilon > 0$  along with a schedule

$$x^\epsilon = (x_1, x_2, \dots, x_{i_0}, x_{i_0+1} + \epsilon, x_{i_0+2}, \dots, x_n)$$

such that  $G_{i_0}(x^\epsilon) < u(x)$  still holds. Since

$$G_{i_0+1}(x^\epsilon) < G_{i_0+1}(x)$$

and

$$G_r(x^\epsilon) = G_r(x), \quad r=0, 1, \dots, i_0-1, i_0+2, \dots, n,$$

$x^\varepsilon$  is also a minimax schedule. By induction, we can find a minimax schedule  $x^*$  such that  $G_n(x^*) < u(x^*)$  which is a contradiction. Hence,  $G_i(x) = u(x)$  for all  $i$ .

Case 2.  $\{j\}_{n \geq j \geq m+1} \neq \emptyset$ .

Case 2. can be proved by three steps:

Step 1. To show  $\phi_0(x) = \dots = \phi_m(x)$   
and  $\psi_{m+1}(x) = \dots = \psi_n(x)$ .

Step 2. To show there exists  $i_0$  and  $j_0$  such  
that  $G_{i_0 m+1}(x) = \dots = G_{i_0 n}(x)$  and  $G_{j_0}(x) = \dots = G_{m j_0}(x)$ .

Step 3. To show  $Z$  is nonnegative.

Step 1. By proposition 2.5.  $U(x) = \sup_{i,j} \{G_{ij}(x)\}$

and  $U(x) = \sup_{m+1 \leq j \leq n} \{\psi_j(x)\}$  follows. By proposition 2.3.

and the definition of  $\psi_j(x)$ , we know  $\psi_j(x)$  is continuous and decreasing on  $x_j$  for  $j = m+2, \dots, n$  and is continuous and increasing on  $x_{j+1}$  for  $j = m+1, \dots, n-1$ . If we treat  $x_{m+1}$  as a fixed initial point then the method in case 1 can show

$$\psi_{m+1}(x) = \dots = \psi_n(x) = u(x).$$

; analogously, if we treat  $x_{m+1}$  as a fixed terminal point we have

$$\phi_0(x) = \dots = \phi_m(x) = u(x).$$

Step 2. Suppose  $\psi_{m+1}(x) = G_{i_0 m+1}(x)$  and  $G_{i_0 m+2}(x) < G_{i_0 m+1}(x)$ .

Since

$$G_{i_0 m+2}(x) = (i_0+1)C + V(x_{i_0+1} - x_{i_0}) + \frac{G_{m+2}(x) - G_{i_0}(x)}{x_{m+2} - x_{i_0}} (\mu - x_{i_0}),$$

$$G_{i_0 m+1}(x) = (i_0+1)C + V(x_{i_0+1} - x_{i_0}) + \frac{G_{m+1}(x) - G_{i_0}(x)}{x_{m+1} - x_{i_0}} (\mu - x_{i_0}).$$

we have

$$G_{i_0 m+2}(x) < G_{i_0 m+1}(x)$$

$$\Leftrightarrow \frac{G_{m+2}(x) - G_{i_0}(x)}{x_{m+2} - x_{i_0}} < \frac{G_{m+1}(x) - G_{i_0}(x)}{x_{m+1} - x_{i_0}}$$

Since  $x_{m+2} - x_{m+1} > 0$  by lemma 2(a), the above inequality

is equivalent to

$$(2.2.14) \quad \frac{G_{m+2}(x) - G_{m+1}(x)}{x_{m+2} - x_{m+1}} < \frac{G_{m+1}(x) - G_{i_0}(x)}{x_{m+1} - x_{i_0}}.$$

At the same time, by the definition of  $\psi_{m+1}(x)$ , we have

$G_{im+1}(x) \leq G_{i_0 m+1}(x)$  for all  $i$ , and since

$$G_{im+1}(x) = G_{m+1}(x) - \frac{G_{m+1}(x) - G_i(x)}{x_{m+1} - x_i} (x_{m+1} - \mu)$$

$$G_{i_0 m+1}(x) = G_{m+1}(x) - \frac{G_{m+1}(x) - G_{i_0}(x)}{x_{m+1} - x_{i_0}} (x_{m+1} - \mu),$$

we have

$$(2.2.15) \quad \frac{G_{m+1}(x) - G_i(x)}{x_{m+1} - x_i} \geq \frac{G_{m+1}(x) - G_{i_0}(x)}{x_{m+1} - x_{i_0}}$$

$$> \frac{G_{m+2}(x) - G_{m+1}(x)}{x_{m+2} - x_{m+1}} \quad (\text{by 2.2.14}).$$

Thus,

$$\begin{aligned} G_{im+2}(x) &= G_i(x) + \frac{G_{m+2}(x) - G_i(x)}{x_{m+2} - x_i} (\mu - x_i) \\ &= G_i(x) + \frac{[G_{m+2}(x) - G_{m+1}(x)] + [G_{m+1}(x) - G_i(x)]}{(x_{m+2} - x_{m+1}) + (x_{m+1} - x_i)} (\mu - x_i). \end{aligned}$$

From (2.2.15), Lemma 2.1(a), and  $x_{m+2} - x_{m+1} > 0$

we have

$$\begin{aligned} G_{im+2}(x) &< G_i(x) + \frac{G_{m+1}(x) - G_i(x)}{x_{m+1} - x_i} (\mu - x_i) \\ &= G_{im+1}(x); \text{ hence,} \end{aligned}$$

$$\begin{aligned} G_{im+2}(x) &< G_{im+1}(x) \leq G_{i_0 m+1}(x) = u(x) \text{ for all } i \\ \Rightarrow \psi_{m+2}(x) &< G_{i_0 m+1}(x) = \psi_{m+1}(x), \text{ a contradiction with} \end{aligned}$$

Step 1. Thus,  $G_{i_0 m+2}(x) = G_{i_0 m+1}(x)$ . By induction we have

$$(2.2.16) \quad G_{i_0 m+1}(x) = G_{i_0 m+2}(x) = \dots = G_{i_0 n}(x).$$

By Lemma 2.1(b) and

$\phi_0(x) = \phi_1(x) = \dots = \phi_m(x) = u(x)$ , we can also prove that there is a  $j_0$  such that (2.2.17)  $G_{0j_0}(x) = G_{1j_0}(x) = \dots = G_{mj_0}(x)$ .

Equations (2.2.16), (2.2.17), and proposition 2.6. implies  $x$  is a SEAI.

Step. 3. If  $z < 0$ , by  $x$  is a SEAI, we then have



$$\begin{aligned}
G_{ij}(x) &= G_i(x) + \frac{G_j(x) - G_i(x)}{x_j - x_i}(\mu - x_i) \\
&= G_i(x) + Z(\mu - x_i) \\
&< G_i(x) \quad \text{for all } (i,j),
\end{aligned}$$

which contradicts  $u(x) = \sup_{i,j} \{G_{ij}(x)\}$ .

Thus, the theorem has been proved.

Theorem 2.3. If  $x$  is a SEAI with non-negative average increment  $Z$  then  $G_0(x) + Z\mu = u(x)$ .

Proof: If  $Z=0$ , by the definition of SEAI, we have

$$G_0(x) = G_1(x) = \dots = G_n(x) \text{ which implies}$$

$$G_{ij}(x) = G_i(x) = G_j(x) = G_0(x) \text{ for all } (i,j); \text{ thus by}$$

theorem 2.1.,

$$\begin{aligned}
u(x) &= \sup_{i,j} \{G_{ij}(x), G_i(x)\} = G_0(x) \\
&= G_0(x) + Z\mu.
\end{aligned}$$

If  $Z > 0$ , by definition of SEAI, we have

$$G_0(x) < G_1(x) < \dots < G_n(x) \text{ which implies } G_{ij}(x) > G_i(x)$$

for all  $(i,j)$ ; thus, by theorem 2.1, we have

$$u(x) = \sup_{i,j} \{G_{ij}(x), G_i(x)\} = \sup_{i,j} \{G_{ij}(x)\}.$$

Because

$$\begin{aligned}
G_{ij}(x) &= G_i(x) + \Delta(j,i)(\mu - x_i) \\
&= G_j(x) - \Delta(j,i)(x_j - \mu) \quad \text{for all } (i,j);
\end{aligned}$$

and

$$\begin{aligned}
 (2.2.18) \quad \Delta(j,i) &= \frac{G_j(x) - G_i(x)}{x_j - x_i} \\
 &= \frac{G_j(x) - G_{j-1}(x) + G_{j-1}(x) - \dots + G_{i+1}(x) - G_i(x)}{\delta_{j-1} + \delta_{j-2} + \dots + \delta_i} \\
 &= \frac{Z[\delta_{j-1} + \delta_{j-2} + \dots + \delta_i]}{\delta_{j-1} + \delta_{j-2} + \dots + \delta_i} = Z,
 \end{aligned}$$

the following hold

$$(2.2.19) \quad G_{im+1}(x) = G_{im+2}(x) = \dots = G_{in}(x) \quad \text{for all } i$$

$$(2.2.20) \quad G_{0j}(x) = G_{1j}(x) = \dots = G_{mj}(x) \quad \text{for all } j.$$

Equation (2.2.19) and (2.2.20) imply all  $G_{i,j}(x)$ 's are equal. This is evident by the following diagram.

$$\begin{array}{ccccccc}
 G_{0 \ m+1}(x) & = & G_{0 \ m+2}(x) & = & \dots & = & G_{0n}(x) \\
 " & & " & & & & " \\
 G_{1 \ m+1}(x) & = & G_{1 \ m+2}(x) & = & \dots & = & G_{1n}(x) \\
 " & & " & & & & " \\
 . & & . & & & & . \\
 . & & . & & & & . \\
 . & & . & & & & . \\
 G_{m \ m+1}(x) & = & G_{m \ m+2}(x) & = & \dots & = & G_{mn}(x)
 \end{array}$$

, where the row equalities follow by (2.2.19) and the column equalities follow by (2.2.20).

By (2.2.18), we have

$$\begin{aligned}
 U(x) &= \sup_{i,j} \{G_{ij}(x)\} = G_{0n}(x) = G_0(x) + \Delta(n,0)\mu \\
 &= G_0(x) + Z\mu
 \end{aligned}$$

Q.E.D.

### 2.3. The properties of objective functions.

Let

$$g(b, n) = c + \frac{bTv - (n+1)c}{(1+b)^{n+1} - 1} + \frac{c}{b} + bvp$$

Proposition 2.7. If  $b^* > 0$ ,  $n^*$  is a positive integer, and

$$g(b^*, n^*) = \min_{(b, n) \in A} g(b, n), \text{ where}$$

$$A = \{(b, n) \mid b \in \mathbb{R}^+, n \in \mathbb{N}, \text{ and}$$

$$(bn - \frac{TV}{c} b^2 - 1) (1+b)^n + 1 < 0\}$$

then

$$\delta_0 = \frac{b^* T - (n^* + 1) \frac{c}{v}}{(1+b^*)^{n^*+1} - 1} + \frac{c}{b^* v},$$

$$\delta_r = (1+b^*)^r \delta_0 - \frac{(1+b^*)^r - 1}{b^*} \frac{c}{v},$$

for  $r=0, 1, \dots, n^*$ , and  $\delta_r = x_{r+1} - x_r$  define the minimax schedule.

Proof. By theorem 2.2. and 2.3., the minimax schedule,  $x^*$ , minimizes

$$G_0(x) + Z\mu$$

where

$$G_{r+1}(x) - G_r(x) = Z\delta_r$$

and

$$\sum_{r=0}^n \delta_r = T \quad \delta_r > 0, Z > 0, r = 0, 1, \dots, n.$$

Equivalently,

$$(2.3.1) \quad C + v\delta_0 + Z\mu$$

is minimized over  $\delta_0$  with constraints

$$(2.3.2) \quad C + v\delta_{r+1} = v\delta_r + Z\delta_r = (v+Z)\delta_r$$

$$(2.3.3) \quad \sum_{r=0}^n \delta_r > 0, \quad Z > 0, \quad r=0,1,\dots,n$$

holding.

However (2.3.2) and (2.3.3)  $\Rightarrow$

$$(2.3.4) \quad \delta_{r+1} = (1+\frac{Z}{v})\delta_r - \frac{C}{v}$$

and

$$\sum_{r=0}^n \delta_r = T, \quad \delta_r > 0, \quad Z > 0, \quad r=0,1,\dots,n.$$

Let  $b = \frac{Z}{v}$ , then from (2.3.4) we have

$$(2.3.5) \quad \delta_r = (1+b)^r \delta_0 - \frac{(1+b)^r - 1}{b} \frac{C}{v}, \quad r=0,1,\dots,n$$

$$\sum_{r=0}^n \delta_r = T, \quad \delta_r > 0, \quad b > 0$$

which implies

$$\delta_0 \sum_{r=0}^n (1+b)^r - \sum_{r=0}^n \frac{(1+b)^r - 1}{b} \frac{C}{v} = T,$$

$$\delta_r > 0, \quad b > 0.$$

Rearranging,

$$(\delta_0 - \frac{C}{bv}) \sum_{r=0}^n (1+b)^r = T - \frac{(n+1)C}{bv}, \quad \delta_r > 0, \quad b > 0,$$

and solving for  $\delta_0$  we have

$$(2.3.6) \quad \delta_0 = \frac{C}{bv} + (T - \frac{(n+1)C}{bv}) \frac{1}{\sum_{r=0}^n (1+b)^r}$$

$$= \frac{C}{bv} + (T - \frac{(n+1)C}{bv}) \frac{b}{(1+b)^{n+1} - 1}$$

$$= \frac{bT - (n+1)\frac{c}{v}}{(1+b)^{n+1} - 1} + \frac{c}{bv}, \quad \delta_r > 0, b > 0.$$

From (2.3.4) in order for  $\delta_{r+1} > 0$ , we have

$$(1 + \frac{z}{v}) \delta_r - \frac{c}{v} > 0; \text{ i.e. } \delta_r > \frac{c/v}{(1+\frac{z}{v})} > 0.$$

Thus, if  $\delta_n > 0$  then  $\delta_0, \delta_1, \dots, \delta_n > 0$ . From (2.3.5) and (2.3.6) we get

$$\delta_n = (1+b)^n \delta_0 - \frac{(1+b)^n - 1}{b} \frac{c}{v}; \text{ hence,}$$

$$\delta_n > 0$$

$$\Leftrightarrow (1+b)^n \delta_0 > \frac{(1+b)^n - 1}{b} \frac{c}{v}$$

$$\Leftrightarrow \delta_0 > \frac{(1+b)^n - 1}{b(1+b)^n} \frac{c}{v}$$

$$\Leftrightarrow \frac{bT - (n+1)\frac{c}{v}}{(1+b)^{n+1} - 1} + \frac{c}{bv} > \frac{(1+b)^n - 1}{b(1+b)^n} \frac{c}{v}$$

$$\Leftrightarrow (nb - \frac{Tv}{c} b^2 - 1) (1+b)^n + 1 < 0.$$

Thus, we get

$$c + v\delta_{r+1} = (v + z)\delta_r$$

$$\sum_{r=0}^n \delta_r = T, \quad \delta_r > 0, \quad z > 0, \quad r=0,1,\dots,n$$

(i.e. (2.3.2) and (2.3.3) holds)

$\Leftrightarrow$

$$\delta_0 = \frac{bT - (n+1)\frac{c}{v}}{(1+b)^{n+1} - 1} + \frac{c}{bv},$$

$$(nb - \frac{Tv}{c} b^2 - 1) (1+b)^n + 1 < 0,$$

$$c + v\delta_{r+1} = (v+Z)\delta_r, \quad r=0,1,\dots,n-1.$$

However,

$$\begin{aligned} & c + v\delta_0 + Z\mu \\ &= c + v\left(\frac{bT - (n+1)\frac{c}{v}}{(1+b)^{n+1} - 1} + \frac{c}{bv}\right) + Z\mu \\ &= g(b,n) \end{aligned}$$

; thus, problem (2.3.1), (2.3.2), (2.3.3) is equivalent to

$\min_{b,n} g(b,n)$ ; subject to

$$(nb - \frac{Tv}{c} b^2 + 1) (1+b)^n + 1 < 0$$

and the associated schedule is

$$(2.3.7) \quad \delta_{r+1} = (1+b)\delta_r - \frac{c}{v} \quad r=1,\dots,n-1,$$

$$\delta_0 = \frac{bT - (n+1)\frac{c}{v}}{(1+b)^{n+1} - 1} + \frac{c}{bv}.$$

Q.E.D.

Proposition 2.8. If  $n(b) = \max\{n \mid (n,b) \in A\}$  then

$$\min_{b \in R^+} g(b, n(b)) = \min_{(b,n) \in A} g(b,n).$$

Proof. On taking difference,

$$\begin{aligned} & g(b,n) - g(b,n-1) \\ &= \frac{c + (1+b)^n (ncb - Tv b^2 - c)}{[(1+b)^{n+1} - 1] [(1+b)^n - 1]}. \end{aligned}$$

Thus,  $g(b,n) - g(b,n-1) < 0$  and  $n' < n$  implies

$$g(b,n') - g(b,n'-1) < 0; \text{ also,}$$

$$g(b,n) - g(b,n-1) > 0 \text{ and } n' > n \text{ implies}$$

$$g(b, n') - g(n'-1) > 0$$

from which the proposition follows.

Since  $(b, n) \in A$  if and only if

$$(nb - \frac{Tv}{c} b^2 - 1)(1+b)^n + 1 < 0, \text{ we define}$$

$$f_n(b) = (nb - \frac{Tv}{c} b^2 - 1)(1+b)^n + 1. \text{ We then have}$$

$$n(b) = \max_{n \in \mathbb{N}} \{n \mid f_n(b) < 0\}.$$

$$\text{Let } n_0 = \lim_{b \rightarrow 0^+} n(b).$$

$$\text{Proposition 2.9. } n_0 \geq \max_{n \in \mathbb{N}} \{n \mid n(n+1) < \frac{2Tv}{c}\}.$$

Proof. Expanding  $(1+b)^n$  we have

$$f_n(b) = \frac{n(n+1)}{2} b^2 - \frac{Tv}{c} b^2 + o(b^2) < 0.$$

Then

$$\frac{n(n+1)}{2} - \frac{Tv}{c} < \frac{o(b^2)}{b^2}, \text{ as } b \rightarrow 0 \text{ and thus}$$

$$n(n+1) \leq \frac{2Tv}{c}.$$

Q.E.D.

The equality in proposition 2.9. holds when  $\frac{2Tv}{c}$  is not an integer, and  $\max_{n \in \mathbb{N}} \{n \mid n(n+1) < \frac{2Tv}{c}\}$  is the optimal number of

inspections in Derman's schedule, the minimax inspection schedule when nothing is assumed about the distribution of  $Y$ .

$$\text{Proposition 2.10. } \lim_{b \rightarrow 0^+} f_n(b) = n(n+1) - \frac{2Tv}{c}.$$

Proof. Follows from the proof of proposition 2.9.

Q.E.D.

From the definition of  $n_0$  and proposition 2.10., we have

$f_{n_0+1}(0^+) \geq 0$ ; thus we can define  $b_{n_0+1}$  as follows:

Definition:  $b_{n_0+1} = \sup_{b>0} \{b \mid f_{n_0}(b) < 0, f_{n_0+1}(b) \geq 0\}$ .

Proposition 2.11.  $b_{n_0+1}$  is the unique zero of  $f_{n_0+1}(\cdot)$  on  $(0, \infty)$ , and  $n(b) = n_0$  for  $0 < b \leq b_{n_0+1}$ .

Proof.

$$(2.3.8) \quad f'_{n_0+1}(b) = b(1+b)^{n_0}[(n_0+1)(n_0+2) - \frac{2Tv}{c} - (n_0+3)\frac{Tv}{c}b].$$

By (2.3.8) we have  $f'_{n_0+1} \downarrow b$  and  $f_{n_0+1}(\infty) = -\infty$  which implies  $f_{n_0+1}(\cdot)$  has an unique zero on  $(0, \infty)$  (see Fig 1).

Let  $f_{n_0+1}(x) = 0$ , we

have

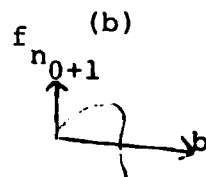


Fig.1.

$$(2.3.9) \quad f_{n_0+1}(b) \geq 0 \quad \text{if } 0 < b \leq x$$

$$(2.3.10) \quad f_{n_0+1}(b) < 0 \quad \text{if } b > x.$$

Also, by proposition 2.9., we have

$$(2.3.11) \quad f'_{n_0}(b) = b(1+b)^{n_0}[n_0(n_0+1) - \frac{2Tv}{c} - (n_0+2)\frac{Tv}{c}b] < 0$$

which implies

$$(2.3.12) \quad f_{n_0}(b) < 0, \quad \text{if } 0 < b \leq x.$$

The definition of  $n(b)$ ,  $b_{n_0+1}$ , (2.3.9), (2.3.10), (2.3.12) prove the proposition.

If  $f_{k+1}(b_{k+1}) = 0$ ,  $f_{k+1}(x) < 0$  for  $x > b_{k+1}$ , and  $f_{k+1}(x) > 0$  for  $0 < x < b_{k+1}$ , then we have

$$f_{k+2}(b_{k+1} + \epsilon) = \{(k+1)b_{k+1} + (k+1)\epsilon + b_{k+1} + \epsilon - \frac{Tv}{c}(b_{k+1} + \epsilon)^2 - 1\} \times (1+b_{k+1} + \epsilon)^{k+1}(1+b_{k+1} + \epsilon) + 1. \quad \text{As } \epsilon \rightarrow 0$$



$$(2.3.13) \quad f_{k+2}(b_{k+1}+) \geq 0$$

holds; also,

$$f_{k+1}(b_{k+1} + \varepsilon) < 0 \quad \forall \varepsilon > 0.$$

We can inductively define  $b_{k+1}$ ,  $k=n_{0+1}, \dots$ , as follows:

Definition:  $b_{k+2} = \sup \{b \mid f_{k+1}(b) < 0, f_{k+2}(b) \geq 0\}$ .

Proposition 2.12. The unique zeros  $b_{n_{0+1}}, \dots$  of  $f_{n_{0+1}}, \dots$ , indicate

$$n(b)=k \text{ if } b_k < b \leq b_{k+1} \text{ where } k=n_0, n_{0+1}, \dots \text{ and } b_{n_0}=0.$$

$$\text{Proof. } f'_{k+2}(b) = b(1+b)^{k+1} \left[ (k+2)(k+3) - \frac{2Tv}{c} - (k+4)\frac{Tv}{c}b \right],$$

$f_{k+2}(\infty) = -\infty$ , and (2.3.13) implies  $f_{k+2}(\cdot)$  has a unique zero on  $(b_{k+1}, \infty)$ . Denote the zero by  $x$ ; then we have

$$f_{k+2}(b) \geq 0 \quad \text{if } b_{k+1} < b \leq x,$$

$$f_{k+2}(b) < 0 \quad \text{if } b > x.$$

Also, from

$$f'_{k+1}(b) < 0 \text{ for } b > b_{k+1}$$

together with

$$f_{k+1}(b_{k+1}) = 0$$

we have

$$f_{k+1}(b) < 0, \quad b_{k+1} < b \leq x.$$

Q.E.D.

Proposition 2.13.  $g(b, n(b))$  is a continuous function.

Proof.

$$\begin{aligned}
 & \lim_{b \rightarrow b_{k+1}} g(b, n(b)) - \lim_{b \rightarrow b_{k+1}} g(b, n(b)) \\
 &= g(b_{k+1}, k+1) - g(b_{k+1}, k) \\
 &= \frac{cf_{k+1}(b_{k+1})}{[(1+b_{k+1})^{k+1} - 1] [(1+b_k)^k - 1]} = 0
 \end{aligned}$$

Q.E.D.

Proposition 2.14. For a given  $b > 0$ , let  $k = n(b)$  and

$$s(b) = \frac{(1+b)^{k+1} - 1}{(1+b)^{k+1}}, \text{ if } \frac{Tv}{c} s(b_{k+1}) \geq 6 \text{ then } g(b, k)$$

is a convex function on  $(b_k, b_{k+1}]$ .

Proof. By taking first and second derivatives on  $g(b, k)$  with respect to  $b$ , we have

$$\begin{aligned}
 (2.3.14) \quad & \frac{d}{db} g(b, k) \\
 &= \left\{ \frac{-\frac{Tv}{c} \{1 + (1+b)^k \{kb - \frac{Tv}{c} (\frac{k+1}{\frac{Tv}{c}})^2 - 1\}\}}{[(1+b)^{k+1} - 1]^2} - \frac{1}{b^2} + \frac{v\mu}{c} \right\} c
 \end{aligned}$$

and

$$\begin{aligned}
 (2.3.15) \quad & \frac{d^2}{db^2} g(b, k) \\
 &= \left\{ \frac{-\frac{Tv}{c} (1+b)^{k-1} \{ (1+b)^{k+1} \phi(b, k) - (k^2 + 3k + 1)(1+b) - k(k+1)(1 + \frac{k+1}{\frac{Tv}{c}}) \}}{\{(1+b)^{k+1} - 1\}^3} \right. \\
 & \quad \left. + \frac{2}{b^3} \right\} c
 \end{aligned}$$

$$, \text{ where } \phi(b, k) = k(k-1)b + 2 - (k-2) \frac{(k+1)^2}{\frac{Tv}{c}}.$$

Let  $x = b_{k+1}$ , by definition, we have

$$1 + (1+x)^{k+1} \left\{ (k+1)x - \frac{Tv}{c} x^2 - 1 \right\} = 0$$

$$\Rightarrow n+1 = \frac{Tv}{c} x + \frac{s(x)}{x}.$$

For convenience, let  $a = \frac{Tv}{c}$  and  $s = s(x)$ , then

$$\begin{aligned} -x^3 \phi(x, k) &= (sa-2)ax^4 - (3s+2)ax^3 + 2as^2x^2 - 3s^2x \\ &+ s^3 = a \left\{ (sa-2)x^2 - (3s+2)x + \frac{25}{16}s^2 \right\} x^2 + s^2 \left\{ \frac{7}{16} ax^2 - 3x + s \right\}. \end{aligned}$$

The discriminant of the quadratic polynomial in the first brace of the above formulation is

$$(2.3.16) \quad (3s+2)^2 - 4(sa-2)\frac{25}{16} < 25 + \frac{200}{16} - sa \frac{100}{16}$$

, and in the second brace is

$$(2.3.17) \quad 9 - \frac{28}{16}as.$$

The condition  $as \geq 6$  makes both (2.3.16) and (2.3.17) negative; hence, we have  $\phi(x, k) < 0$ . Since  $\phi(\cdot, k) \uparrow b$  for  $b_k < b < b_{n+1} = x$ , we then have  $\frac{d^2}{db^2} g(b, k) > 0$  for  $b \in (b_k, b_{k+1}]$ .

Q.E.D.

Proposition 2.15.  $\frac{d}{db} g(b, n(b)) > 0$  if  $b > \sqrt{\frac{c}{uv}}$ .

Proof.  $b > \sqrt{\frac{c}{uv}}$

$$(2.3.18) \Rightarrow -\frac{1}{b^2} + \frac{vu}{c} > 0$$

Also,  $f_{n+1}(\frac{n+1}{\frac{Tv}{c}})$

$$= 1 + \left(1 + \frac{n+1}{\frac{Tv}{c}}\right)^{n+1} \left\{ \frac{(n+1)^2}{\frac{Tv}{c}} - \frac{Tv}{c} \left(\frac{n+1}{\frac{Tv}{c}}\right)^2 - 1 \right\}$$

$$= 1 + \left(1 + \frac{n+1}{\frac{Tv}{c}}\right)^{n+1} (-1) < 0$$

which implies  $\frac{n+1}{\frac{Tv}{c}} > b_{n+1} > b$

$$; \text{ thus, } 1 + (1+b)^n \left\{ nb - \frac{Tv}{c} \left(\frac{n+1}{\frac{Tv}{c}}\right)^2 - 1 \right\}$$

$$< 1 + (1+b)^n \left\{ nb - \frac{Tv}{c} b^2 - 1 \right\}$$

$$= f_n(b) < 0, \text{ where } n = n(b).$$

Together with (2.3.18) and (2.3.14), we have

$$\frac{d}{db} g(b, n(b)) > 0.$$

Q.E.D.

Proposition 2.16.  $\frac{Tv}{c} s(b) \uparrow_b$ .

Proof. Recall that  $s(b) = \frac{(1+b_{n+1})^{n+1} - 1}{(1+b_{n+1})^{n+1}}$ , where

$n = n(b)$ . By proposition 2.12., we have  $n(b) \uparrow_b$  and

$$b_{n+1} \uparrow_b.$$

Q.E.D.

As a summary of proposition 2.13, 2.14, 2.15 and 2.16, we have the following theorem.

Theorem 2.4. If  $\frac{Tv}{c} s(0+) \geq 6$ , then the objective function  $g(b, n(b))$  is continuous and piecewise convex, and the optimal  $b^*$  is in  $[0, \sqrt{\frac{c}{v\mu}})$ .

Proof. By proposition 2.13., 2.14., 2.15., 2.16..

Q.E.D.

2.4. The asymptotic minimax schedule as  $T \rightarrow \infty$ .

Definitions:  $X_T$  denotes a schedule on  $[0, T]$ .

For a given schedule  $x$ ,  $n(x)$  is the total number of inspections assigned,  $\delta_i(x)$  is the length between  $i$ th and  $i+1$ th inspections, and  $u(x)$  is the corresponding supremum expected cost. If  $x$  is a SEAI then  $Z(x)$  is its average increment level. If  $\delta_i(x) = \frac{T}{n(x)+1}$  for all  $i$ , then  $x$  is called a strictly periodic schedule. If  $s$  is a real number,  $[s]$  denotes the largest integer which is small than  $s$ .

Proposition 2.17. If  $x$  is a strictly periodic schedule then  $x$  is a SEAI.

$$\text{Proof. } \Delta(i+1, i) = \frac{(i+2)c + v\delta_{i+1}(x) - (i+1)c - v\delta_i(x)}{\delta_i(x)}$$

$$= \frac{c(n(x)+1)}{T}.$$

Q.E.D.

Proposition 2.18. If  $\{T_j\}$  is a time sequence and  $T_j \rightarrow \infty$  as  $j \rightarrow \infty$ , then  $n(x_{T_j}^*) \rightarrow \infty$  as  $j \rightarrow \infty$ , where  $x_{T_j}^*$  is the minimax schedule on  $[0, T_j]$ .

Proof. Suppose  $x_{T_j}^0$  is Derman's schedule on  $[0, T_j]$

then  $Z(x_{T_j}^0) = 0$ , by proposition 2.12., we have

$$n(x_{T_j}^0) \leq n(x_{T_j}^*).$$

But  $\lim_{j \rightarrow \infty} n(x_{T_j}^0) = \infty$ . The proposition follows.

Theorem 2.5. If  $x_T^*$  is the minimax schedule on  $[0, T]$  then there exists a strictly periodic schedule  $x_T^0$  for each  $T$  such that

$$\begin{aligned} \lim_{T \rightarrow \infty} u(x_T^0) &= \lim_{T \rightarrow \infty} u(x_T^*) \\ &= c + v \sqrt{\frac{\mu c}{v}} + \mu \sqrt{\frac{\mu c}{v}}. \end{aligned}$$

Proof. By theorem 2.3.,  $x_T^*$  is a SEAI and

$$(2.4.1) \quad u(x_T^*) = c + v \delta_0(x_T^*) + Z(x_T^*) \mu.$$

Let  $\{T_j\}$  be any time sequence such that  $T_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $x_{T_j}^0$  be a strictly periodic schedule such that

$$\begin{aligned} 1 + n(x_{T_j}^0) &= \left\lceil \frac{T_j}{\sqrt{\frac{\mu c}{v}}} \right\rceil \\ u(x_{T_j}^0) &= c + v \frac{T_j}{\frac{T_j}{\sqrt{\frac{\mu c}{v}}}} + \mu \frac{\left\lceil \frac{T_j}{\sqrt{\frac{\mu c}{v}}} \right\rceil}{T_j}. \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \frac{\left\lceil \frac{T_j}{\sqrt{\frac{\mu c}{v}}} \right\rceil}{T_j} = \frac{1}{\sqrt{\frac{\mu c}{v}}} \quad \text{and} \quad u(x_{T_j}^*) \leq u(x_{T_j}^0),$$

we obtain

$$(2.4.2) \quad \lim_{j \rightarrow \infty} u(x_{T_j}^*) \leq c + \sqrt{\frac{vc}{v}} + \mu \frac{c}{\sqrt{\frac{vc}{v}}} < \infty.$$

(2.4.1) and (2.4.2) imply that  $\limsup_{j \rightarrow \infty} \delta_0(x_{T_j}^*)$

and  $\limsup_{j \rightarrow \infty} Z(x_{T_j}^*)$  are both finite.

Suppose 0 is a limit point of  $\{Z(x_{T_j}^*)\}$

and  $\lim_{j_k \rightarrow \infty} Z(x_{T_{j_k}}^*) = 0$ , where  $\{T_{j_k}\}$  is a subsequence of  $\{T_j\}$ .

By proposition 2.7. and  $b = \frac{Z}{v}$ , we have

$$\begin{aligned} & \lim_{j_k \rightarrow \infty} u(x_{T_{j_k}}^*) \\ &= \lim_{j_k \rightarrow \infty} c + \frac{TZ(x_{T_{j_k}}^*) - (n(x_{T_{j_k}}^*) + 1)c}{(1 + Z(x_{T_{j_k}}^*))^{n(x_{T_{j_k}}^*) + 1} - 1} + \frac{vc}{Z(x_{T_{j_k}}^*)} \\ & \quad + Z(x_{T_{j_k}}^*)\mu \\ &= \lim_{n \rightarrow \infty} \lim_{Z \rightarrow 0} c + \frac{TZ^2 - (n+1)cZ + vc[(1 + \frac{Z}{v})^{n+1} - 1]}{Z[(1 + \frac{Z}{v})^{n+1} - 1]} + Z\mu. \end{aligned}$$

Applying L'Hospital's rule twice, we obtain

$$\lim_{Z \rightarrow 0} \frac{TZ^2 - (n+1)cZ + vc[(1 + \frac{Z}{v})^{n+1} - 1]}{Z[(1 + \frac{Z}{v})^{n+1} - 1]} = \frac{Tv}{n+1} + \frac{nc}{Z}$$

which implies

$$\lim_{j_k \rightarrow \infty} u(x_{T_{j_k}}^*) = \lim_{n \rightarrow \infty} \frac{(n+1)c}{2} + \frac{Tv}{n+1} = \infty$$

, a contradiction with (2.4.2). Thus,

$$(2.4.3) \quad \lim_{j_k \rightarrow \infty} z(x_{T_{j_k}}^*) = z^0 > 0.$$

By proposition 2.7. we have

$$(2.4.4) \quad \delta_0(x_{T_{j_k}}^*) = \frac{z(x_{T_{j_k}}^*)^{\frac{T}{v}} - (n(x_{T_{j_k}}^*) + 1)\frac{c}{v}}{(1+z(x_{T_{j_k}}^*))^{n(x_{T_{j_k}}^*) + 1} - 1} + \frac{c}{z(x_{T_{j_k}}^*)}.$$

Because (2.4.3) holds it can be seen that the first term of the right-hand side of (2.3.3) tends to zero when  $j_k \rightarrow \infty$ ; thus,  $\lim_{j_k \rightarrow \infty} \delta_0(x_{T_{j_k}}^*) = \frac{c}{z^0}$ ,

$$\text{Since } c + v\frac{c}{z^0} + \mu z^0 \geq c + v\sqrt{\frac{\mu c}{v}} + \mu\sqrt{\frac{c}{\mu c/v}} \text{ is always true}$$

for  $z^0 > 0$ , we have

$$\begin{aligned} & \lim_{j_k \rightarrow \infty} u(x_{T_{j_k}}^*) \\ &= \lim_{j_k \rightarrow \infty} c + v\delta_0(x_{T_{j_k}}^*) + \mu z(x_{T_{j_k}}^*) \\ &= c + v\frac{c}{z^0} + \mu z^0 \\ &\geq c + v\sqrt{\frac{\mu c}{v}} + \mu\sqrt{\frac{c}{\mu c/v}}, \end{aligned}$$



together with (2.4.2) and the definition of  $\{x_{T_j}^0\}$ , we have

$$\lim_{j_k \rightarrow \infty} u(x_{T_{j_k}}^*) = c + v \sqrt{\frac{\mu c}{v}} + \mu \frac{c}{\sqrt{\frac{\mu c}{v}}} = \lim_{j \rightarrow \infty} u(x_{T_j}^*)$$

Q.E.D.

## 2.5. Optimization by an algorithm.

The objective  $\min_{b \geq 0} g(b, n(b))$  in proposition 2.8. can be

standerlized as follows.

$$\begin{aligned} & \min_{b \geq 0} g(b, n(b)) \\ &= C \min_{b \geq 0} \left\{ 1 + \frac{b \frac{Tv}{c} - [n(b) + 1]}{(1+b)^{n(b)+1} - 1} + \frac{1}{b} + b \frac{\mu}{T} \frac{Tv}{c} \right\} \\ &= C \min_{b \geq 0} \bar{g}(b, n(b), \frac{Tv}{c}, \frac{\mu}{T}) \end{aligned}$$

, and by multiplying  $\delta_i$ 's by  $\frac{v}{c}$  respectively, (2.3.6) becomes

$$\bar{\delta}_0(b) = \frac{v}{c} \delta_0 = \frac{b \frac{Tv}{c} - [n(b) + 1]}{(1+b)^{n(b)+1} - 1} + \frac{1}{b}$$

and

$$\bar{\delta}_r(b) = \frac{v}{c} \delta_r = (1+b)^r \bar{\delta}_0(b) - \frac{(1+b)^r - 1}{b}$$

for  $r=1, 2, \dots, n(b)$ .

Note that  $\bar{g}$  and  $\bar{\delta}_r(b)$ 's are determined by  $\frac{Tv}{c}$ ,  $\frac{\mu}{T}$  and

$b$ , and the optimal  $\bar{b}^*$  for  $\bar{g}$  is also optimal for  $g$ .

For the optimal  $b^*$  we have following algorithm.

Algorithm 2.1.

Step 1. Use the bisection method to obtain the unique zero of  $f_{n_0+1}, f_{n_0+2}, \dots$  on  $[0, \infty), [b_{n_0+1}, \infty), [b_{n_0+2}, \infty), \dots$ , respectively.

If  $b_{k+1} > \sqrt{\frac{c}{11V}}$  then  $\bar{n}=k$  and go to step 2.

Step 2. If  $\frac{TV}{c} s(b_{n_0+1}) \geq 6$ , go to step 3; otherwise, let  $m = \max \{k : \frac{TV}{c} s(b_{k+1}) < 6, n_0 < k \leq \bar{n}\}$ , go to step 4.

Step 3. If  $\frac{d\bar{g}}{db}(b_{n_0+1}, n_0) > 0$  then use the bisection method to obtain the unique zero  $\tilde{b}_{n_0}$  of  $\frac{d\bar{g}}{db}(\cdot, n_0)$  in  $[\varepsilon, b_{n_0+1}]$ , where  $\varepsilon$  is a small positive number such that  $\frac{d\bar{g}}{db}(\varepsilon, n_0) < 0$ . If  $\frac{d\bar{g}}{db}(b_{n_0+1}, n_0) \leq 0$ , let  $\tilde{b}_{n_0} = b_{n_0} + 1$ . Set  $m = n_0$  then go to step 5.

Step 4. Graph  $g(b, n(b))$  from 0 to  $bm+1$ , let  $\tilde{b}_m$  satisfy  $g(\tilde{b}_m) = \min_{0 \leq b \leq b_{m+1}} \{g(b, n(b))\}$ . Go to step 5.

Step 5. Let  $k=m, \dots, \bar{n}$ .

If  $\frac{d\bar{g}}{db}(b_k, k) \frac{d\bar{g}}{db}(b_{k+1}, k) < 0$  then use the bisection method to obtain the unique zero  $\tilde{b}_k$  of  $\frac{d\bar{g}}{db}(\cdot, k)$  in  $[d_k, b_{k+1}]$ . If  $\frac{d\bar{g}}{db}(b_k, k) \frac{d\bar{g}}{db}(b_{k+1}, k) \geq 0$  and  $\frac{d\bar{g}}{db}(b_k, k) \geq 0$  then  $\tilde{b}_k = b_k$ . If  $\frac{d\bar{g}}{db}(b_k, k) \frac{d\bar{g}}{db}(b_{k+1}, k) \geq 0$  and

$\frac{d\bar{g}}{db}(b_k, k) < 0$  then  $\tilde{b}_k = b_{k+1}$ . Cf. T of step 6.

Step 6. The optimal  $b^*$  satisfies

$$q(b^*, (n(b^*))) = \min_{k=m, \dots, \bar{n}} \{q(\tilde{b}_k, n(\tilde{b}_k))\}.$$

The finiteness of step 1.

Suppose  $\lim_{k \rightarrow \infty} b_k < \sqrt{\frac{c}{\mu\nu}}$ , by the definition of  $b_k$  and

proposition 2.11 we have

$$(1 + \sqrt{\frac{c}{\mu\nu}})^k (k \sqrt{\frac{c}{\mu\nu}} - \frac{T\nu}{c} \frac{c}{\mu\nu} + 1) < 0 \quad \text{as } k \rightarrow \infty$$

which is false. Hence, there is a  $\bar{n}+1$  such that  $b_{\bar{n}+1}^- > \sqrt{\frac{c}{\mu\nu}}$ .

The optimization of algorithm 2.1. is then followed by the finiteness of step 1., theorem 2.4, proposition 2.7 and proposition 2.8.

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